J. Fluid Mech. (1973), *vol.* 60, *part* 4, *pp.* 813–824
Printed in Great Britain

On an asymptotic solution of the Korteweg–de Vries equation with slowly varying coefficients

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(Received 12 March 1973)

The variable-coefficient Korteweg–de Vries equation

$$H_X + \tfrac{3}{2}d^{-\frac{7}{4}}HH_\xi + \tfrac{1}{6}\kappa d^{\frac{1}{2}}H_{\xi\xi\xi} = 0$$

with $d = d(\epsilon X)$ is discussed for solitary-wave initial profiles. A straightforward asymptotic solution for $\epsilon \to 0$ is constructed and is shown to be non-uniform both ahead of and behind the solitary wave. The behaviour ahead is rectified by matching to the appropriate exponential form and, together with the use of conservation laws for the equation, the nature of the solution behind the solitary wave is discussed. This leads to the formulation of the solution in the oscillatory 'tail', which is again matched directly.

The results are applied to the development of the solitary wave into variable-depth water, and the predictions are compared with those obtained, for example, by Grimshaw (1970, 1971). Finally, the asymptotic behaviour of both the solitary wave and the oscillatory tail are assessed in the light of some numerical integrations of the equation.

1. Introduction

With the upsurge of interest and success in the solution of nonlinear wave problems, which has occurred over the last decade or so, has come the Korteweg–de Vries (K–dV) equation in particular. Although this equation cannot represent the breaking of waves, it does enable smooth profiles to be formed by balancing the nonlinearity against dispersive effects. The equation may be written in the form

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

where $u(x, t)$ is proportional to the amplitude of the wave, and (1) readily admits steady-state waves $u(x-ct)$, which are described by the Jacobian elliptic cn-function. The special non-periodic limit of the oscillatory 'cnoidal' wave is the well-known solitary wave.

This equation was first derived by Korteweg & de Vries (1895) to describe shallow-water waves in a two-dimensional channel of constant depth. More recently, the same equation has also been found to apply, for example, to collisionless plasmas (e.g. Kakutani *et al.* 1968). The real interest in the K–dV equation has sprung from the exact solutions found by Miura, Gardener & Kruskal (1968) which describe the production of a finite number of solitary waves (or 'solitons') from appropriate initial data. In particular, certain initial profiles of the form $a\,\text{sech}^2(\alpha x)$ will eventually decompose into an integral number of solitons.

In 1969, Madsen & Mei reported that when a solitary wave, which was unchanging in a depth of water of, say, unity, entered a region where the bottom of the channel shelved, solitons were formed. This was predicted numerically by integrating equations based on the Boussinesq approximation and reasonable agreement was found with experimental data. Tappert & Zabusky (1971) showed, by using a WKB approximation for an abrupt change in depth, that a constant-coefficient K–dV equation could be constructed which gave the number (and amplitudes) of the solitons on the shelf. If the bottom shelved to a new constant depth $d_0\,(<1)$, then just n solitons would eventually appear along the shelf provided that

$$d_0 = [\tfrac{1}{2}n(n+1)]^{-\frac{4}{9}}. \tag{2}$$

This result assumes that wave reflexion from the step may be neglected.

The same prediction, (2), was also found by Johnson (1973); however, the procedure adopted here was to construct the relevant *variable*-coefficient K–dV equation. This was accomplished by deriving the K–dV equation in the standard (asymptotic) formulation but with the depth allowed to vary slowly on the scale of the appropriate small parameter. If, after non-dimensionalization, the amplitude is described by the parameter δ and the dispersive effects by (depth/wavelength)$^2 = \kappa\delta\ (\kappa = O(1))$, then an asymptotic solution for $\delta \to 0$ may be derived. In the far field (time or distance $O(\delta^{-1})$, linearized characteristic co-ordinate $O(1)$) the constant-coefficient K–dV equation (1) gives the relevant description of the surface profile. Now, if the depth of the channel is allowed to vary slowly as a function of the (non-dimensional) distance z, then $d = d(\delta\epsilon z)$, where ϵ is another parameter (which may be unity). With the use of the full linearized characteristic co-ordinate

$$\xi = \int_0^z d^{-\frac{1}{2}}\,dz - t = O(1)$$

and the far-field co-ordinate $X = \delta z$, we obtain

$$2d^{\frac{1}{2}}\eta_X + \tfrac{1}{2}(d_X/d^{\frac{1}{2}})\,\eta + (3/d)\,\eta\eta_\xi + \tfrac{1}{3}\kappa\, d\eta_{\xi\xi\xi} = 0 \tag{3}$$

(a result also given by Ostrovskiy & Pelinovskiy (1970)†), which is valid in the far field as $\delta \to 0$. The disturbance of the surface level from unity is $\delta\eta(X,\xi)$. Removal of the 'Green's Law' attenuation factor

$$\eta = d^{-\frac{1}{4}}H(X,\xi)$$

gives the variable-coefficient K–dV equation

$$H_X + \tfrac{3}{2}d^{-\frac{7}{4}}HH_\xi + \tfrac{1}{6}\kappa d^{\frac{1}{2}}H_{\xi\xi\xi} = 0 \quad (d = d(\epsilon X)). \tag{4}$$

A discussion of this equation has previously been given by Johnson (1973), and some numerical results were presented in Johnson (1972). In the latter paper, the depth was changed not too slowly (e.g. $\epsilon = 1$) and the soliton 'fission' was reproduced. The agreement with Madsen & Mei (1969) was surprisingly good. In

† The author is grateful to a referee for indicating the existence of this paper.

Johnson (1973) it was proved that if the depth changes rapidly enough in (4) then the solitons will be produced according to (2) (for appropriate initial data). However, the problem of the ultra-slowly varying depth (i.e. $\epsilon \to 0$) was not examined.

The purpose of the present paper is to construct an asymptotic solution to (4), as $\epsilon \to 0$, with a solitary-wave initial condition. This initial solitary wave will be just a steady-state solution of (4) with $d = 1$. As an overall aim, we shall first attempt to describe the development of the solitary wave. The results will be compared with a corresponding study undertaken by Grimshaw (1970, 1971), where interest was centred on the more complete Boussinesq equations. In fact, some of Grimshaw's conclusions seem a little in doubt owing to the lack of uniform validity of his asymptotic expansions (see Ablowitz 1971). Second, and perhaps of even more interest, is the analysis of the structure of the oscillatory tail behind the solitary wave. In this particular problem it is comforting to find that the oscillatory nature conforms with an elementary intuitive approach, e.g. approximating the nonlinear term. However, the formal construction and matching of this solution must be treated with a little care.

2. Perturbation of the solitary wave

Using the interpretation of the variable-coefficient K–dV equation (4) that has already been introduced, the variable X plays the role of a time-like coordinate. Thus the initial-value problem may be described as follows. At $X = 0$ (and physically for all $X < 0$), the surface profile is a solitary wave which moves with unchanging form over a depth $d(0) = 1$. For $X \geqslant 0$ the depth smoothly changes. Also

$$H(X, \xi) \to 0 \quad \text{as} \quad \xi \to \pm\infty \quad \text{for all} \quad X < \infty.$$

Now, since the depth slowly changes by virtue of the parameter ϵ, it seems most convenient to seek a solution which is essentially a perturbation of the initial solitary wave. However, since the development of the wave will presumably occur on the scale ϵX, we introduce this as a 'slow' variable together with a quasi-steady-state variable T:

$$\epsilon X = x, \quad T = \xi - c(x)\,X, \tag{5}$$

where $c(x)$ is to be found. It is an unnecessary complication also to introduce a slow ξ (or T) variable at this stage.

Substituting the new variables (5) into (4) gives

$$\epsilon H_x - (cx)'\,H_T + \tfrac{3}{2}d^{-\frac{7}{4}}HH_T + \tfrac{1}{6}\kappa d^{\frac{1}{2}}H_{TTT} = 0 \quad (d = d(x)), \tag{6}$$

where the prime denotes differentiation with respect to x. Seeking an asymptotic solution in the form

$$H(x, T; \epsilon) = H_0(x, T) + \epsilon H_1(x, T) + o(\epsilon), \tag{7}$$

where
$$H_0 = b(x) + a(x)\,\mathrm{sech}^2\,[\alpha(x)\,T], \tag{8}$$

implies that
$$\alpha^2 = (3/4\kappa)\,d^{-\frac{9}{4}}a, \quad (cx)' = \tfrac{1}{2}d^{-\frac{7}{4}}(a + 3b). \tag{9}$$

If zero subscripts denote initial values, then the initial solitary wave is described by

$$\alpha_0 = (3a_0/4\kappa)^{\frac{1}{2}}, \quad c_0 = \tfrac{1}{2}(a_0+3b_0) \tag{10}$$

and a_0 and b_0 define the family of all such solitary waves. The relations (9) are deduced from the zeroth-order equation for H_0 by requiring that (8) be a valid description for all T.

The equation for H_1 becomes

$$-(cx)' H_{1T} + \tfrac{3}{2}d^{-\frac{7}{4}}(H_0 H_1)_T + \tfrac{1}{6}\kappa d^{\frac{1}{2}} H_{1TTT} = -H_{0x}, \tag{11}$$

which may be integrated directly (by writing $H_1 = H_{0T} f(x, T)$) to give

$$H_1 = \frac{-6}{\kappa d^{\frac{1}{2}}} H_{0T}\left[\int^T \frac{1}{H_{0T}^2}\left\{\int^T H_{0T}\left(\int^T H_{0x}\, dT + A\right) dT + B\right\} dT + C\right]. \tag{12}$$

$A(x)$, $B(x)$ and $C(x)$ are arbitrary functions to be chosen by applying appropriate boundary conditions to $H(x, T; \epsilon)$ (and hence H_1).

Ahead of the solitary wave ($T \to +\infty$) the flow is undisturbed and consequently we choose A, B and C (and perhaps other functions) so that $H_1 \to 0$ as rapidly as possible in this limit. To this end, it is clear that the terms in (12) arising from $b(x)$ will give growing terms as $T \to +\infty$ (of the form $T \sinh 2\alpha T$) unless

$$db/dx = 0.$$

Thus $b(x) \equiv b_0$, and without loss of generality we may choose $b_0 = 0$. This result is to be expected since we may always consider the solitary wave to be above the undisturbed level. Now to find A, B and C it is necessary to construct the solution for H_1, and then examine its behaviour as $T \to +\infty$. Of course, we could obtain the relevant information by simpler means but we shall require a rather complete knowledge of H_1 to enable the details of the matching to be successfully completed. Thus writing H_0 in (12) (with $b \equiv 0$) and integrating yields

$$\begin{aligned} H_1 = \frac{3}{\kappa d^{\frac{1}{2}}}\frac{1}{a\alpha^2} \operatorname{sech}^2 \alpha T \tanh \alpha T \Bigg[&\left\{ a\left(\frac{a}{\alpha}\right)' - \frac{1}{6}\left(\frac{a^2}{\alpha}\right)' \right\} (\ln|\sinh \alpha T| + \tfrac{1}{4}\cosh 2\alpha T) \\ &+ \tfrac{1}{2}(a^2/\alpha^2)\, \alpha' \{\tfrac{1}{2}\alpha^2 T^2 - \alpha T \coth \alpha T + \ln|\sinh \alpha T|\} \\ &+ \tfrac{1}{4} a A \{6\alpha T - 4\coth \alpha T + \sinh 2\alpha T\} \\ &- \tfrac{1}{3}(a^2/\alpha)' \{\ln|\sinh \alpha T| + \sinh^2 \alpha T + \tfrac{1}{4}\sinh^4 \alpha T\} \\ &+ \tfrac{1}{32} B \{60\alpha T - 32\coth \alpha T + 16 \sinh 2\alpha T + \sinh 4\alpha T\} + C \Bigg]. \end{aligned} \tag{13}$$

It is now an elementary exercise to examine (13) and it is evident that the original expansion (7) will be non-uniform in both the limits $T \to \pm\infty$ (i.e. H_1/H_0 is unbounded). To remove the exponential growth in H_1 as $T \to +\infty$, we choose

$$B = \frac{1}{3}\left(\frac{a^2}{\alpha}\right)', \quad A = \frac{1}{6a}\left(\frac{a^2}{\alpha}\right)' - \left(\frac{a}{\alpha}\right)' \tag{14}$$

and $C(x)$ cannot be defined. In fact C may be carried through and defined, together with two of the three arbitrary functions which arise later, to ensure that

H_2 is similarly behaved. Thus $C(x)$ plays the role in H_2 that $b(x)$ played here in H_1.

Using the chosen form for $A(x)$ and $B(x)$, we observe that the expansion is still algebraically non-uniform as $T \to +\infty$, but exponentially so as $T \to -\infty$†. It would also appear that certainly the algebraic difficulties would be removable if we were to introduce an appropriate 'slow'-T variable in the original formulation. Of course, this would give rise to a set of partial differential equations defining a, b, α and c, together with an even more complicated expression for H_1. The exponential non-uniformity as $T \to -\infty$ may have to be treated rather more circumspectly.

Now, to describe the solitary wave, the three functions a, α and c must be defined. So far the analysis has produced only two equations relating them, equations (9), and consequently the solution is not unique. It is to be hoped that the procedure of matching to the appropriate solution outside the range of validity of (7) will completely determine $H_0(x, T)$.

3. Matching ahead of the solitary wave

The algebraic breakdown of the asymptotic expansion as $T \to +\infty$ suggests that the appropriate form for H is

$$H \sim \left(\sum_{m=0}^{\infty} \epsilon^m \beta_m(x) T^m\right) \exp\left[\sum_{n=1}^{\infty} \gamma_n(x)\, \epsilon^{n-1} T^n\right] + \mathscr{E}. \tag{15}$$

$\mathscr{E}$ denotes terms exponentially small compared with those retained and these will arise from the nonlinear term in (6). Now (15) can be more conveniently written by introducing a 'slow'-T variable, as already suggested:

$$H \sim \left(\sum_{n=0}^{\infty} \epsilon^n h_n(x, \tau)\right) \exp\left[\frac{1}{\epsilon} f(x, \tau)\right] + \mathscr{E}, \quad \epsilon T = \tau. \tag{16}$$

The introduction of the functions $h_n(x, \tau)$ enables (16) to be a generalization of (15). However, we expect that if (16) were rewritten in terms of T, and expanded for $\epsilon \to 0$, it would agree with (15), which in turn should match with (7) (as $T \to +\infty$). Thus the solution (16) is to be matched directly to the original expansion.

To construct the expansion (16), equation (6) is rewritten with $\tau = \epsilon T$:

$$H_x - (cx)' H_\tau + \tfrac{3}{2} d^{-\frac{7}{4}} H H_\tau + \tfrac{1}{6}\epsilon^2 \kappa d^{\frac{1}{2}} H_{\tau\tau\tau} = 0, \tag{17}$$

and substituting from (16) yields

$$f_x - (cx)' f_\tau + \tfrac{1}{6}\kappa d^{\frac{1}{2}} f_\tau^3 = 0, \tag{18}$$

$$h_{0x} - (cx)' h_{0\tau} + \tfrac{1}{2}\kappa d^{\frac{1}{2}}[h_0 f_\tau f_{\tau\tau} + h_{0\tau} f_\tau^2] = 0, \tag{19}$$

and so on. Unfortunately it is not possible to write down a simple representation of the general solution for $f(x, \tau)$, although this solution may be obtained by

† Note that any difficulty arising from the behaviour of $d(x)$ (e.g. $d(x) \to 0$) is excluded from the present analysis.

standard techniques. For the purposes of the present study it is most convenient to seek a solution as $\tau \to 0$ of the form

$$f(x,\tau) \sim \alpha_1(x)\,\tau + \alpha_2(x)\,\tau^2 + \ldots .$$

This gives directly

$$\alpha_1(x) = \pm\left[\frac{6(cx)'}{\kappa d^{\frac{1}{2}}}\right]^{\frac{1}{2}} = \pm\, 2\alpha(x), \quad \alpha_2(x) = \frac{\alpha_1'}{2(cx)' - \kappa d^{\frac{1}{2}}\alpha_1^2},$$

and correspondingly for h_0,

$$h_0 \sim \beta_0(x) + \frac{2\beta_0' + 2\kappa d^{\frac{1}{2}}\alpha_1\alpha_2\beta_0}{2(cx)' - \kappa d^{\frac{1}{2}}\alpha_1^2}\tau + \ldots,$$

where $\beta_0(x)$ is arbitrary. Thus the behaviour of $H(x,\tau)$ as $\tau \to 0$ becomes

$$H(x,\tau) \sim \beta_0 e^{\pm(1/\epsilon)2\alpha\tau}\left[1 + \left(\frac{3\alpha'\beta_0 - 2\alpha\beta_0'}{4\beta_0(cx)'}\right)\tau \mp \frac{1}{2}\frac{\alpha'}{[\epsilon(cx)']}\tau^2\right] \quad (\tau = \epsilon T), \tag{20}$$

and the expansion of $H(x,T)$, from (7), (8), (9) and (13), as $T \to +\infty$ is

$$H(x,T) \sim 4ae^{-2\alpha T}\left[1 + \epsilon\left(\frac{1}{2}\frac{\alpha'}{(cx)'}T^2 + \frac{a'}{a(cx)'}T\right)\right]. \tag{21}$$

The two expansions, (20) and (21), match exactly if we

$$\left.\begin{array}{ll}\text{(i)} & \text{choose the negative sign for } \alpha_1(x),\\ \text{(ii)} & \text{choose } \beta_0(x) = 4a(x),\\ \text{(iii)} & \text{choose } (a^2/\alpha)' = 0.\end{array}\right\} \tag{22}$$

Now (iii) is the one final condition that was required to enable the slowly developing solitary wave to be completely defined. As to its interpretation, we shall see shortly that it is related to the momentum distribution in the complete wave profile. Before considering this point in detail, the behaviour far ahead of the solitary wave will be examined, i.e. $\tau \to +\infty$.

The general solution for $f(x,\tau)$ can easily be shown to exhibit similarity properties as $\tau \to +\infty$. In particular, the similarity solution of (18) is

$$f(x,\tau) = \pm\tfrac{2}{3}[\zeta/D^{\frac{1}{3}}]^{\frac{3}{2}} \quad (\zeta = \epsilon\xi = \tau + cx), \tag{23}$$

where
$$D(x) = \tfrac{1}{2}\kappa\int_0^x d^{\frac{1}{2}}\,dx.$$

The negative sign is appropriate if the profile is to decay. Solution (23) is exactly the exponent in the asymptotic form of the Airy function as deduced by solving the linearized form of (17). To find the asymptotic behaviour of $h_0(x,\tau)$, it is convenient to use the exact solution of (19), which (with the aid of (18)) is

$$h_0(x,\tau) = f_{\zeta\zeta}^{\frac{1}{2}}\bar{h}_0(f_\zeta), \tag{24}$$

where $\bar{h}_0$ is an arbitrary function. The matching condition

$$h_0(x,0) = 4a(x)$$

(which has already been employed in (22)) enables h_0 to be completely determined. If we introduce the similarity form for $f(x, \tau)$, then a little manipulation yields

$$h_0 \sim k_0 D^{-\frac{1}{2}} \tag{25}$$

as $\zeta \to +\infty$, where k_0 is a prescribed constant depending on the initial solitary wave and the particular depth variation. Solution (25) is one of the family of similarity solutions of (19), but is not related to those of the constant-coefficient K–dV equation (see appendix).

The behaviour of the profile far ahead of the solitary wave is thus described by

$$H(x, \tau) \sim k_0 D^{-\frac{1}{2}} \exp\left[-\frac{1}{\epsilon}\frac{2}{3}\left(\frac{\tau + cx}{D^{\frac{1}{3}}}\right)^{\frac{3}{2}}\right], \tag{26}$$

which decays in both the limits

$$x \text{ fixed}, \quad \tau \to \infty; \quad \tau \text{ fixed}, \quad x \to \infty.$$

It has been assumed here that $d(x)$ either approaches a new constant depth or steadily increases as $x \to \infty$. The 'shoreline' problem denoted by $d \to 0$ has already been excluded.

Finally, to complete this section, the matching condition (22) is used to determine the solitary wave, so that

$$\frac{a}{a_0} = d^{-\frac{3}{4}}, \quad \frac{\alpha}{\alpha_0} = d^{-\frac{3}{2}}, \quad \frac{c}{c_0} = \frac{1}{x}\int_0^x d^{-\frac{5}{2}}\, dx. \tag{27}$$

Note that the observed solitary-wave amplitude is given by

$$a d^{-\frac{1}{4}} = a_0 d^{-1} \tag{28}$$

when the Green's law attenuation is reintroduced.

4. The oscillatory tail

We have already noted that the straightforward expansion is exponentially non-uniform as $T \to -\infty$. In particular (8), (13) and (14) give

$$H \sim 4a\, e^{2\alpha T} - \epsilon \frac{3}{\kappa d^{\frac{1}{2}}} \frac{1}{\alpha^2}\left(\frac{a}{\alpha}\right)' \quad (T \to -\infty), \tag{29}$$

so that the expansion becomes invalid when $2\alpha T = O(\ln \epsilon)$ or $H = O(\epsilon)$. This suggests that the profile behind the solitary wave may be uniformly $O(\epsilon)$, which, if correct, will greatly simplify the construction of a solution. To confirm this, and also to make some general comments on the structure of the oscillations, we can argue as follows.

The original equation (4) (or alternatively (6)) yields two simple conservation laws. [Although the standard K–dV equation yields an infinity of such laws, the variable-coefficient equation has only two;† see Johnson (1973).] By using the fact that $H \to 0$ for $\xi \to \pm\infty$ (for all X, $x < \infty$), we obtain

$$\int_{-\infty}^{\infty} H(x, T; \epsilon)\, dT = \text{const}, \quad \int_{-\infty}^{\infty} H^2(x, T; \epsilon)\, dT = \text{const}, \tag{30}$$

† There is one other conservation law which can easily be written down; however, it is not 'simple' in that x and ξ appear explicitly. It corresponds to the special law given by Miura *et al.* (1968).

which are just the conservation of mass and momentum, respectively. Note that (30) are exact. The constants may be evaluated by introducing the initial forms for H at $x = 0$. This yields

$$\int_{-\infty}^{\infty} H(x,T;\epsilon)\,dT = \frac{2a_0}{\alpha_0}, \quad \int_{-\infty}^{\infty} H^2(x,T;\epsilon)\,dT = \frac{4}{3}\frac{a_0^2}{\alpha_0}.$$

Now for $T = O(1)$ we have an asymptotic representation of the solution to $O(\epsilon)$, and as $T \to +\infty$ it differs only in exponentially small terms from the behaviour of the sech^2 function. We may thus find an estimate for the mass and momentum carried by the profile behind the solitary wave.

Let $T_0(<0)$ be some $O(1)$ value of T which may be regarded as defining the interface between the expansion we have (for $T = O(1)$) and the solution we do not yet have for $T \to -\infty$. That this demarcation is not explicit in practice is irrelevant since our aim is to produce an order-of-magnitude estimate only. Writing the range of integration as $(-\infty, T_0)$ and (T_0, ∞) and using the known solution in the latter integrals gives

$$\int_W H\,dT = 2\left(\frac{a_0}{\alpha_0} - \frac{a}{\alpha}\right) + O(\epsilon) = 2\left(\frac{a_0}{\alpha_0}\right)(1 - d^{\frac{3}{4}}) + O(\epsilon), \tag{31}$$

$$\int_W H^2\,dT = O(\epsilon) \quad \left(\text{since}\,\frac{a^2}{\alpha} = \frac{a_0^2}{\alpha_0}\right), \tag{32}$$

where W denotes $(-\infty, T_0)$. The error $O(\epsilon)$ in (31) and (32) comes directly from the asymptotic expansion (7), and a further error which is exponentially small (as $\epsilon \to 0$) will arise from the behaviour as $T, \tau \to \infty$.

If the profile in W is of amplitude $\Delta(\epsilon)$ over a range $L(\epsilon)$, outside which the solution is exponentially small, then (31) and (32) imply that

$$\Delta L = O(1), \quad \Delta^2 L = O(\epsilon).$$

Consequently the amplitude in W is necessarily nowhere greater than $O(\epsilon)$.

To implement this information, it is convenient to rewrite (29) as

$$H \sim \epsilon\left[4a\exp\{2\alpha T - \ln\epsilon\} - \frac{3}{\kappa d^{\frac{1}{2}}}\frac{1}{\alpha^2}\left(\frac{a}{\alpha}\right)'\right], \tag{33}$$

where the next term takes the form $\epsilon\exp\{-(2\alpha T - \ln\epsilon)\}$. Now introducing the new variables

$$\epsilon T = \tau, \quad \theta = \epsilon^{-1}F(x,\tau;\epsilon), \quad H = \epsilon\tilde{h}(x,\tau,\theta;\epsilon),\dagger$$

where θ is a phase function, we ensure that any oscillatory behaviour has a constant period in θ. This formulation now corresponds directly with the solution found in §3 for $T \to +\infty$.

Expanding $\tilde{h} = \tilde{h}_0 + \epsilon\tilde{h}_1 + \ldots$, and substituting into (17) yields

$$(F_x - (cx)'F_\tau)\,\tilde{h}_{0\theta} + \tfrac{1}{6}\kappa d^{\frac{1}{2}}F_\tau^3\tilde{h}_{0\theta\theta\theta} = 0,$$

whence for $\tilde{h}_0$ to have a constant period

$$\tilde{h}_0 = A_0(x,\tau) + B_0(x,\tau)\,e^{i\theta} + B_0^*(x,\tau)\,e^{-i\theta} \tag{34}$$

with

$$F_x - (cx)'F_\tau = \tfrac{1}{6}\kappa d^{\frac{1}{2}}F_\tau^3. \tag{35}$$

† *Note added in proof.* In a recent private communication, Dr R. Grimshaw has suggested another formulation equivalent to the one employed here.

Of course, (35) is just (18) written for $\zeta < 0$; however, an important difference arises by virtue of the matching condition which involves ϵ, namely

$$F \sim i(\epsilon \ln \epsilon - 2\alpha\tau) \quad \text{as} \quad \tau \to 0.$$

The equation for $\tilde{h}_1$ becomes

$$\tfrac{1}{6}\kappa d^{\frac{1}{2}} F_\tau^3(\tilde{h}_{1\theta} + \tilde{h}_{1\theta\theta\theta}) + \tilde{h}_{0x} - (cx)'\,\tilde{h}_{0\tau} + \tfrac{3}{2} d^{-\frac{7}{4}} F_\tau \tilde{h}_0 \tilde{h}_{0\theta} + \tfrac{1}{2}\kappa d^{\frac{1}{2}} F_\tau (F_\tau \tilde{h}_{0\theta\theta})_\tau = 0,$$

and for $\tilde{h}_1$ to be periodic in θ, the terms giving rise to secular behaviour must be removed, so that

$$A_{0x} - (cx)'\,A_{0\tau} = 0 \tag{36}$$

and

$$B_{0x} - (cx)'\,B_{0\tau} - \tfrac{1}{2}\kappa d^{\frac{1}{2}} F_\tau (F_\tau B_0)_\tau + i\tfrac{3}{2} d^{-\frac{7}{4}} F_\tau A_0 B_0 = 0. \tag{37}$$

$\tilde{h}_1$ is now forced by terms $e^{\pm 2i\theta}$.

Both (36) and (37) may be integrated to give

$$A_0 = \bar{A}_0(\zeta), \quad B_0 = \bar{B}_0(\psi) \exp\left[\int_0^x \{\tfrac{1}{2}\kappa d^{\frac{1}{2}} \psi F_{\zeta\zeta} - i\tfrac{3}{2} d^{-\frac{7}{4}} \bar{A}_0(\zeta)\,\psi\}|_{\zeta=\bar{F}}\, dx\right], \tag{38}$$

where $\psi = F_\zeta$, so that $\zeta = \bar{F}(\psi, x)$, with $\bar{A}_0$ and $\bar{B}_0$ arbitrary functions. Note that the solution for B_0 involves A_0, arising from the nonlinearity; this is to be compared with h_0; see (24). The matching conditions

$$A_0(x,0) = -\frac{3}{\kappa d^{\frac{1}{2}}}\frac{1}{\alpha^2}\left(\frac{a}{\alpha}\right)', \quad B_0(x,0) = 4a(x)$$

enable A_0 and B_0 to be found completely. As before, it is instructive to examine the solution far behind the solitary wave ($\tau \to -\infty$), and in particular derive the envelope of the oscillations.

Using the similarity solution

$$F = \tfrac{2}{3}(-\zeta/D^{\frac{1}{3}})^{\frac{3}{2}} \quad (\zeta < 0) \quad \text{(see (23))},$$

then (38), after some manipulation, yields

$$A_0 \sim 0, \quad B_0 \sim k_1 D^3 (-\zeta)^{-\frac{7}{2}} \quad (\zeta \to -\infty),$$

where k_1 is a prescribed constant (similar to k_0). Once again, this similarity solution bears no relation to those obtainable from the constant-coefficient K–dV equation.

5. Discussion

Qualitatively, the solution we have obtained for $\epsilon \to 0$ indicates that the soliton retains its identity, as is to be expected. In particular it has been defined by applying the conservation of momentum for the soliton alone, which arose quite naturally as the matching condition across the non-uniformity ahead. Quantitatively, the soliton (as observed) will have an amplitude proportional to d^{-1}, which agrees with Grimshaw (1970, 1971) for small amplitudes. These results also confirm the contention of Ostrovskiy & Pelinovskiy (1970) that this dependence is on d^{-1} and not $d^{-\frac{4}{3}}$.

The phase function αT can also be seen to agree with Grimshaw's analysis. For example, we could suppose that when

$$x \geqslant x_1, \quad d = d_1 \text{ (constant)}$$

(where d_1 is arbitrary but finite and non-zero) then

$$T = \epsilon^{-1}\left[\int_0^{x_1} (\delta^{-1}d^{-\frac{1}{2}} - \tfrac{1}{2}a_0 d^{-\frac{5}{2}})\,dx\right] + \epsilon^{-1}(\delta^{-1}d_1^{-\frac{1}{2}} - \tfrac{1}{2}a_0 d_1^{-\frac{5}{2}})\,(x - x_1) - t.$$

The first term is just a phase shift and so we have a propagation speed

$$d_1^{\frac{1}{2}}(1 - \tfrac{1}{2}\delta a_0 d_1^{-2})^{-1}$$

in the z, t plane. Clearly, as $\delta \to 0$, the classical result is retrieved and higher order terms coincide with Grimshaw (1970). Of course, since Grimshaw assumed that the solitary wave was a uniform representation of the solution, his conclusions will essentially be reproduced by the present theory. Note, however, that Grimshaw studied the more complete Boussinesq equations from which the K–dV equation may be derived.

In § 4, by examining the non-uniformities behind the solitary wave, the solution in the oscillatory tail was constructed. It is worthy of note that although this wave is uniformly $O(\epsilon)$, the mass carried by it may vary by an $O(1)$ amount (see (31)). In fact, if $1 \geqslant d > 0$, the mass transfers to the oscillations as d decreases (being careful to avoid the formal limit $d \to 0$). As d increases from unity the mass in W becomes negative and the amplitude of the soliton decreases. Eventually, say when $d = O(\epsilon^{-1})$, the soliton will no longer be discernible separate from the oscillations and a completely oscillatory profile would seem a reasonable *representation*. This was the asymptotic solution assumed in Johnson (1973) for the development into deeper water.

In conclusion, the efficacy of the asymptotic predictions may be examined by comparing them with some numerical results. Using the scheme discussed in Johnson (1972), the variable-coefficient K–dV equation (4) was integrated for various ϵ and for both increasing and decreasing depths. In figure 1 is given the solitary-wave amplitude for various ϵ in a decreasing depth of water; for comparison the predicted behaviour $d^{-\frac{3}{4}}$ is also shown (the attenuation $d^{-\frac{1}{4}}$ is not included). Second, in figure 2 is shown the peak (and trough) envelope for the oscillatory wave over increasing and decreasing depths, together with the asymptotic behaviour $\zeta^{-\frac{7}{2}}$.

The evidence seems fairly conclusive. First, the amplitude variation clearly approaches the asymptotic solution as $\epsilon \to 0$. Curves for decreasing depth are presented, but the same holds true if the depth is increased. Even more comforting is the close correspondence between the envelope of the oscillatory wave and the power-law decay of $-\frac{7}{2}$. Note that this behaviour is at variance with that predicted by using the constant-coefficient K–dV equation (see appendix).

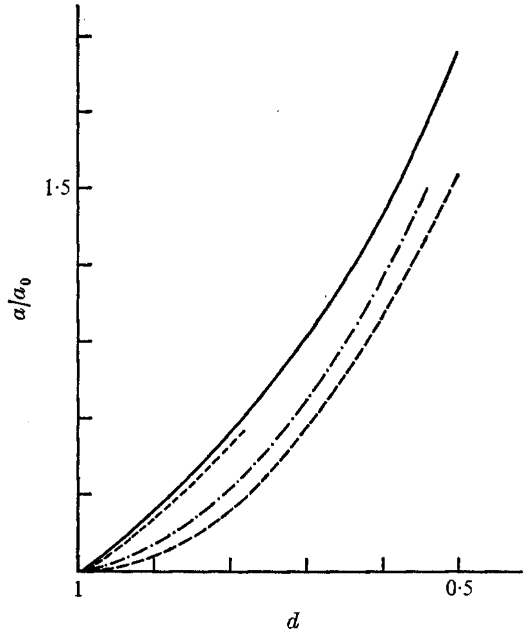

FIGURE 1. The solitary-wave amplitude a/a_0 as obtained from numerical integration of equation (4) for various ϵ. — —, $\epsilon = 0{\cdot}8$; —·—, $\epsilon = 0{\cdot}5$; - - - -, $\epsilon = 0{\cdot}05$. ——, asymptotic theory, $a/a_0 = d^{-\frac{3}{4}}$.

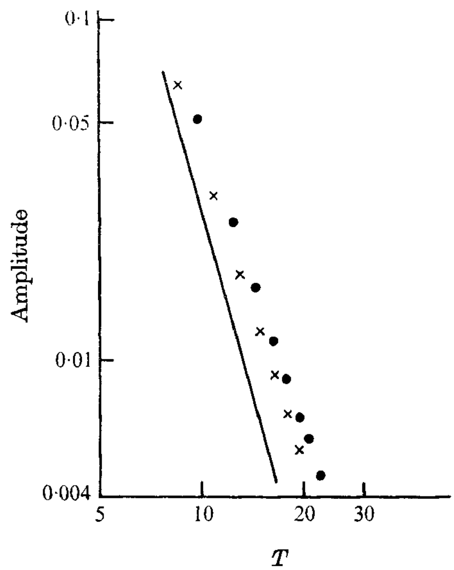

FIGURE 2. The position and amplitude of the peaks and troughs (× − 1) from numerical integrations of equation (4). ●, depth decreasing, $\epsilon = 0{\cdot}05$; ×, depth increasing, $\epsilon = 0{\cdot}1$; ——, theoretical prediction for slope of $-3\frac{1}{2}$. The slope of numerical results is about $-3{\cdot}2$.

Appendix. Similarity solutions of the K–dV equation

The constant-coefficient equation is of the form

$$u_t + uu_x + u_{xxx} = 0,$$

which remains unaltered under the transformation

$$x \to r^{\frac{1}{3}}x, \quad t \to rt, \quad u \to r^{-\frac{2}{3}}u$$

for constant r. Thus a similarity solution may be represented by

$$u = t^p x^{-2-3p} f(\eta), \quad \eta = xt^{-\frac{1}{3}},$$

with the two extreme cases

$$p = 0, \quad u = x^{-2}f(\eta); \quad p = -\tfrac{2}{3}, \quad u = t^{-\frac{2}{3}}f(\eta).$$

The particular value $p = -\frac{3}{4}$ with

$$f(\eta) \text{ exponential as } \eta \to +\infty, \quad f(\eta) \text{ trigonometric as } \eta \to -\infty$$

was discussed by Berezin & Karpman (1964). In a paper shortly to appear in *J. Math. Phys.*, M. J. Ablowitz & A. C. Newell have shown how this similarity solution is relevant to the continuous spectrum of the constant-coefficient K–dV equation.

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